Computers & Structures Vol. 8, pp. 633-640

0045-7949/78/0515-0633/$02.00/0

A HIGH PRECISION TRIANGULAR LAMINATED ANISOTROPIC CYLINDRICAL SHELL FINITE ELEMENT

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(*Received* 1 *February* 1976)

Abstract—The stiffness matrix for a high precision triangular laminated anisotropic cylindrical shell finite element has been formulated and coded into a composite structural analysis program. The versatility of the element's formulation enables its use in the analysis of multilayered composite plate and cylindrical shell type structures taking into account actual lamination parameters. The example applications presented demonstrated that accurate predictions of stresses as well as displacements are obtained with modest number of elements.

INTRODUCTION

Multilayered composite materials are increasingly used in the construction of plate and shell type structures for various industrial and aero-space applications, creating a parallel need for structural analysis procedures which reflect the laminated anisotropic characteristics of such materials. Although the basic equations which govern the elastic behaviour of laminated anisotropic plate and shell structures can be derived[1], analytical solutions of these complex equations, as they pertain to design and analysis, are not available. The analysis of composite structures is an area which calls for an extensive use of digital computers and numerical methods like the finite element method.

The finite element method has proved to be an extremely powerful tool for the solution of problems involving complex geometries, arbitrary loadings and rather general material properties. Considerable advances have been made in the development and application of the finite element method to plate and shell type structures. Much of the work is reviewed by Gallagher[2, 3]. According to these reviews, the analysis of thin shells istone of the more difficult problems that has been attempted with the finite element method. The requirements upon valid minimum potential energy solutions in thin shell finite element analysis are extremely difficult to satisfy, and the satisfactory formulations are therefore relatively complicated. Further, for tackling problems involving description of arbitrary boundaries triangular and quadrilateral elements are obviously best suited, The most reliable and sophisticated triangular shell element formulations are those due to Dupuis[4], Cowper *et al.*[5], Argyris and Scharph[6] and Dawe[7]. The formulation due to Cowper *et al.* [5] has also been converted into a triangular cylindrical shell finite element by Lindberg and Olson [8]. The salient features of this element are: (i) The element is fully conforming, of the displacement type and of arbitrary triangular shape, (ii) Uses higher order interpolation polynomials and has 36 degrees of freedom, (iii) Transverse displacement function contains a complete quartic polynomial plus some higher degree terms and allows cubic variation of normal slope along each edge. (iv) Tangential displacement functions are complete cubic polynomials to yield a quadratic variation of stresses within the element, (v) Gives accurate predictions of displacements as well as stresses, (vi) Satisfies sufficient conditions to guarantee rapid convergence, (vii) Permits reliable and direct evaluation of stresses at nodes, (viii) The element stiffness matrix is formulated in exact closed form, (ix) Results in a smaller overall problem size and (x) The versatility of the formulation enables its use in the analysis of flat plates in either membrane or flexural behaviour and for the coupled membrane-flexural behaviour of cylindrical shells. The formulations of this element as given in Ref. [8] is confined to homogeneous isotropic materials. The present work is concerned with the generalization of this formulation to take into account the laminated anisotropic characteristics of multilayered composite materials.

The purpose of this paper is to present the formulation and some useful applications of a high precision triangular laminated anisotropic cylindrical shell finite element. In order to perform composite structural analysis, a computer program incorporating this element has been developed. Distinct membrane and flexural stiffnesses are retained permitting analysis of composite structures subjected to inplane and bending loads. The applications presented encompassing fiat plates in plane stress and in flexure and circular cylindrical shells clearly indicate the usefulness of the element in composite structural analysis.

FORMULATION

The geometry of an arbitrary triangular cylindrical shell element is shown in Fig. 1. The global cylindrical coordinate system is X, Y and ξ, η are taken as local coordinates for the element. R is the radius of curvature of the reference surface of the shell and t is its wall thickness. The dimesnions a, b, c of the triangle 1, 2, 3 and the rotation angle 6 are easily derived in terms of the global coordinates of the vertices [9].

The strain energy of a thin cylindrical shell in global coordinates is given by

$$U=\frac{1}{2}\int\int\{[N]^T[\epsilon^0]+[M]^T[\chi]\}\,dx\,dy \tag{1}$$

where

$[N]^T=(N_X, N_Y, NXY)$ are the membrane forces
$[M]^T=(M_X, M_Y, MXY)$ are the bending stress resultants
$[\epsilon^0]^T=(\epsilon_X^0, \epsilon_y^0, \gamma_{XY}^0)$ are the reference surface strains and
$[\chi]^T=(\chi_X, \chi_Y, \chi_{XY})$ are the bending curvatures.

For an arbitrarily laminated anisotropic shell, the con-

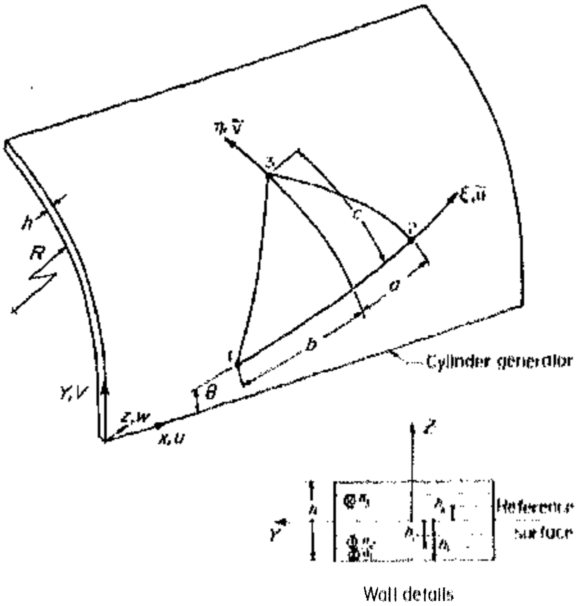


Fig. 1. Cylindrical shell triangular finite element Geometry and coordinate systems,

stitutive relations are expressible in the form[1]

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \epsilon^0 \\ \chi \end{bmatrix} \tag{2}$$

where $[A]$, $[B]$, $[D]$ are the membrane, interaction and bending stiffness matrices of the laminate.

Substituting eqns (2) in (1), we get

$$U = \frac{1}{2} \int\int \{[\epsilon^0]^T [A][\epsilon^0] + [\chi]^T [B][\epsilon^0] + [\epsilon^0]^T [B][\chi] + [\chi]^T [D][\chi]\} \, dx \, dy. \tag{3}$$

Using Novozhilov's thin shell theory, the strain-displacement relations are [10]

$$[\epsilon^0] = \begin{bmatrix} U_X \\ V_Y + W/R \\ U_Y + V_X \end{bmatrix} \tag{4}$$

$$[X] = \begin{bmatrix} -W_{XX} \\ -(W_{YY} - V_{Y/R}) \\ -2(W_{XY} - V_{X/R}) \end{bmatrix} \tag{5}$$

where U, V, W are the displacement components in the global coordinate system (X, Y, Z), W being positive when radially outward. Note that subscript! 1 eqns (4) and (5) denote differentiation.

Substituting for $[\epsilon^0]$ and $[\chi]$ from eqns (4) id (5), it is possible to express the strain energy given I eqn (3) in matrix form as

$$U = \frac{1}{2} \int\int \{[d]^T [L][d]\} \, dx \, dy \tag{6}$$

where

$$[d]^T = (U_X, U_Y, V_X, V_Y, W, W_{XX}, W_{XY}, W_{YY})$$

and the strain energy matrix $[L]$ is given in ' ble 1.

If $\bar{U}$, V denote reference surface displace ents in the local coordinate system (ξ, η), the transfor ation from local to global coordinates is

$$[d] = [R_4][\bar{d}] \tag{7}$$

where

$$[\bar{d}]^T = (\bar{U}_\xi, \bar{U}_\eta, \bar{V}_\xi, \bar{V}_\eta, W, W_{\xi\xi}, W_{\xi\eta}, W_{\eta\eta})$$

1 fc ix $[R_4]$ T le 2. Substituting eqn (7) in eqn (6), the strain e gy of the element is given by

$$U = \frac{1}{2} \int\int \{[\bar{d}]^T [\bar{L}][\bar{d}]\} \, d\xi \, d\eta \tag{8}$$

where

$$[\bar{L}] = [R_4]^T [L][R_4]$$

with the area integration to be carried ove he triangle 1 2 3.

The assumed displacement functions for the element are [5]

$$\bar{U} = \sum_{i=1}^{10} a_i \xi^{m_i} \eta^{n_i}$$

$$\bar{V} = \sum_{i=11}^{20} a_i \xi^{p_i} \eta^{q_i}$$

$$W = \sum_{i=21} a_i \xi^{r_i} \eta^{s_i} \tag{9}$$

where m_i, n_i, p_i, r_i and s_i are just integers[5

The element stiffness matrix is obtained rom a calculation of its strain energy. The displacem t functions of eqns (9) are substituted into eqn (8) and tl integration curried out to yield

$$U = \frac{1}{2} [\alpha]^T [k][\alpha] \tag{10}$$

Table I. Strain energy matrix $[L]$

A_{11}	A_{11}	$A_{11} + \frac{2B_{11}}{R}$	$A_{12} + \frac{B_{12}}{R}$	A_{12}/R	B_{11}	$2B_{11}$	B_{12}
	A_{33}	$A_{33} + \frac{2B_{33}}{R}$	$A_{23} + \frac{B_{23}}{R}$	A_{23}/R	B_{13}	$2B_{33}$	B_{23}
		$A_{33} + \frac{4B_{33}}{R} + \frac{4D_{33}}{R^2}$	$A_{23} + \frac{3B_{23}}{R} + \frac{2D_{23}}{R^2}$	$\frac{A_{23}}{R} + \frac{2B_{23}}{R^2}$	$B_{13} - \frac{2D_{13}}{R}$	$2B_{33} - \frac{4D_{33}}{R}$	$-\frac{2D_{23}}{R}$
			$A_{22} + \frac{2B_{22}}{R} + \frac{D_{22}}{R^2}$	$\frac{A_{22}}{R} + \frac{B_{22}}{R^2}$	$-B_{12} - \frac{D_{12}}{R}$	$-2B_{23} - \frac{2D_{23}}{R}$	$-\frac{D_{22}}{R}$
				A_{22}/R^2	$-B_{12}/R$	$-2B_{23}/R$	B_{22}/R
					D_{11}	$2D_{13}$	D_{12}
		Symmetric				$4D_{33}$	DM
							D_{22}

Table 2. Rotation matrix $[R_4]$

$$\begin{bmatrix} CC & -SC & -SC & SS & 0 & 0 & 0 & 0 \\ SC & CC & -SS & -SC & 0 & 0 & 0 & 0 \\ SC & -SS & CC & -SC & 0 & 0 & 0 & 0 \\ SS & SC & SC & CC & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & & CC & -2SC & SS \\ & & & & & SC & CC-SS & -SC \\ & & & & & SS & 2SC & CC \end{bmatrix}$$

where

$CC = \cos^2\theta \quad SS = \sin^2\theta$

$SC = \sin\theta\cos\theta$

where $[a]$ is a 40-column vector of the polynomial coefficients a_i. The entries of the stiffness matrix $[k]$ are determined in closed form and are given in the Appendix.

The generalized displacements in global coordinates chosen for the element are

$$[W]^T = (U_1, U_{X1}, U_{Y1}, V_1, V_{X1}, V_{Y1}, W_1, W_{X1}, W_{Y1}, W_{XX1}, W_{XY1}, W_{YY1}, U_2 \ldots, U_3, \ldots, W_{YY3}). \quad (11)$$

Further steps to be followed in the deviation of the 36 x 36 stiffness matrix relating the generalized displacements $[W]$ and the corresponding generalized forces, and consistent load vector corresponding to uniform pressure load are very similar to those for the shallow shell element [5].

COMPUTER PROGRAM

The stiffness matrix and consistent load vector for the high precision triangular laminated anisotropic cylindrical shell finite element has been coded into a composite structural analysis program. Distinct membrane and flexural stiffnesses are retained permitting analysis of composite structures subjected to inplane and bending loads. The resulting force-displacement equations are solved using Choleski square root decomposition method for fixed bandwidth problems. Stress resultants and individual layer stresses are directly calculated at the nodes using the generalized displacements, constitutive relations and appropriate transformation equations. With basic lamina elastic constants and laminate details as input, the program evaluates the laminate stiffnesses for each element. It is therefore possible to analyse laminates of varying thickness taking into account the actual lamination parameters. The computer output consists of generalized displacements, stress resultants, membrane and bending strains, individual layer stresses with reference to both laminate and lamina reference axes at each node. With this information it is possible to make strength prediction of practical composite structures using appropriate failure criteria [1].

APPLICATIONS

The versatility of the element formulation enables its use in the analysis of laminated anisotropic flat plates in either membrane or flexural behaviour and for coupled membrane-bending behaviour of circular cylindrical shells. The following example applications illustrate the usefulness of the element in composite structural analysis. All calculations are carried out on an IBM 370/155 machine using double precision arithmetic.

Boron-epoxy composite laminate with a circular hole under axial tension

In Fig. 2 is shown a boron-epoxy composite panel consisting of nine plies of $[0/\pm 45/0/90]_s$ laid up with a central circular hole under uniaxial tension. Symmetry of the loading, geometry and material properties make the analysis of only one quarter of the panel sufficient. The finite element idealization of the quarter panel illustrated in Fig. 3 uses 51 elements with 37 nodes resulting in 186 degrees of freedom. Typical stress distributions calculated at the nodes are shown in Figs. 4 and 5 along with the results of anisotropic elasticity solution to the problem of an infinite plate with a circular opening [11]. The present results appear to have converged quite rapidly. The predicted tensile and compressive stress and strain concentration factors given in Table 3 together with results of anisotropic elasticity solution for the plate clearly demonstrate that the element gives excellent accuracy.

An identical problem has been analysed in Ref. [12] using linear-strain triangular elements (twelve degrees of freedom). One hundred and forty one elements with 320 nodes (598 degrees of freedom) were used to describe

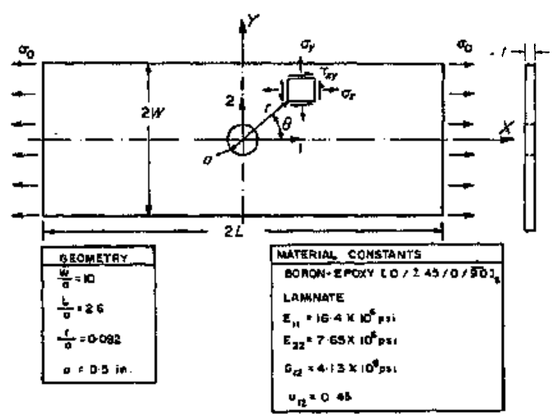


Fig. 2. Boron-epoxy laminate with a circular hole under axial tension.

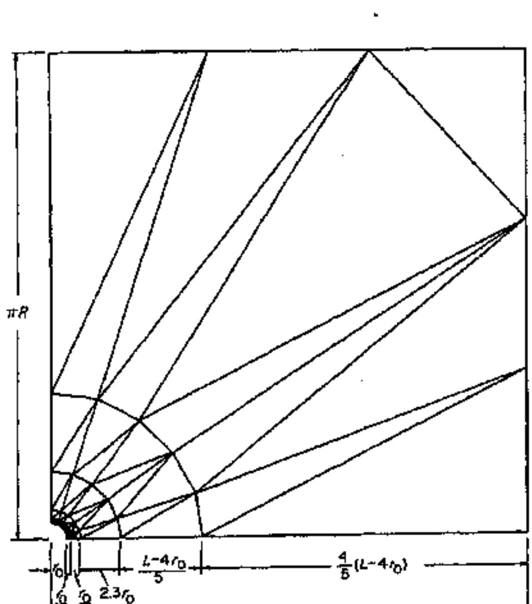


Fig. 3. 5 x 5 Finate element grid.

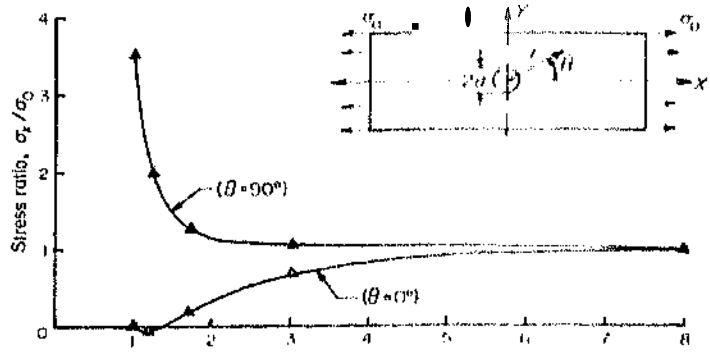


Fig. 4, Axial stress distribution along X and Y axes. , Analytical solution; $\triangle$, finite element resul

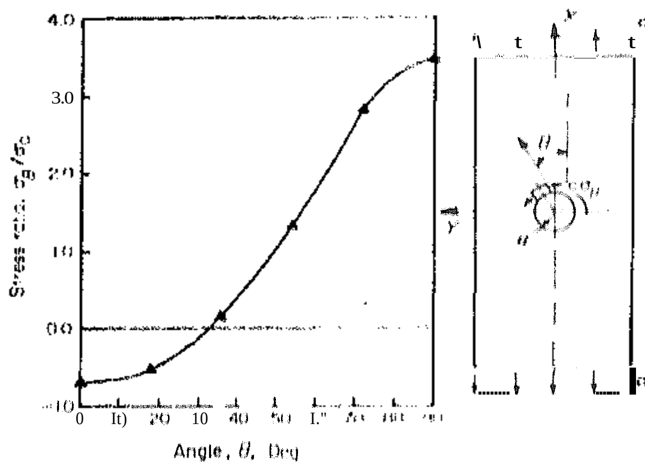


Fig. 5. Tangential stress distribution along hole boundary. , Analytical solution; $\triangle$, finite element re ts.

the quarter panel. The predictions are also given in Table 3 for comparison. The fact that the present formulation yields better accuracy with about one-third the problem size demonstrate the superiority of the element.

The problem has also been analysed experimentally using straingages, photoelastic coatings and moire lech- niques in Ref. (12J. Average predictions are given in Table 3. The close agreement between the present results and experimental data readily verifies the adequacy of the assumed laminated anisotropic model in the analysis as u first approximation to the actual multilayered com- posite material behaviour.

Clamped glass-epoxy composite square plate subjected to uniform pressure load. Figure 6 shows a glass-epoxy composite square plate with clamped edges subjected to uniform pressure load. Symmetry allows the analysis to be limited to one quarter of the plate. Arrangement of the elements in a quadrant of the plate are : o shown in Fig. 6. Predicted deflection distributions ong centr lines of the plate are shown in Fig. 7 al g with the results from an approximate analytical soh on given in Ref, [13]. This analytical solution assu s identical deflection profiles along the centrelines the plate. Whereas, as it should be, the present ana sis predicts different deflection profiles parallel and pe ndicular to the fibre direction. This prediction is i qualitative agreement with experimental observations n unidirec- tional composite plates in flexure using holographic techniques [14]. Bending moment distributi s along the centrelines of the plate shown in Fig. H are irly smooth and can be considered to have converged well. Typi- cal results given in Table 4 can be con lered more accurate than the solution available in[13]. owever, an assessment of actual accuracy of the pred ed stresses

Table 3, Boron-epoxy laminate with a circtilnr hole under axial tension Comparison of results

	No. of unknowns	Stress concentration factors at θ 0°	θ 90°	Strain concentration factors at θ 0°	90°
Present results	186	0.6957	3.4982	1.4725	3.449
Analytical solution. Ref. [12]	—	0,6829	3.45	1.46	3.45
Finite element results. Ref [9]	598	—	3.60	1.58	3.60
Experimental results. Ref. [9]	—	—	3.34	1.5	3.34

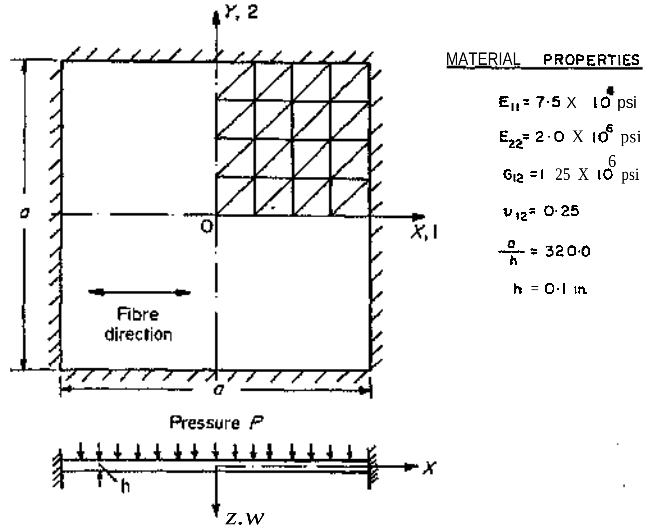


Fig. 6. Finite element grid for clamped glass-epoxy composite square plate.

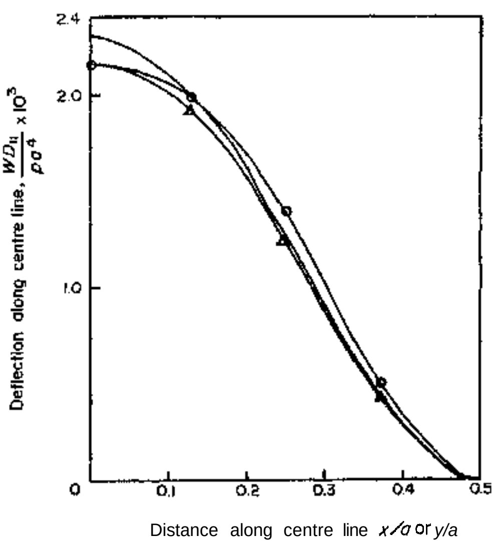


Fig. 7. Deflections of clamped glass-epoxy composite square plate. ——, Analytical solutions; Δ, along X-axis; 0, along Y-axis.

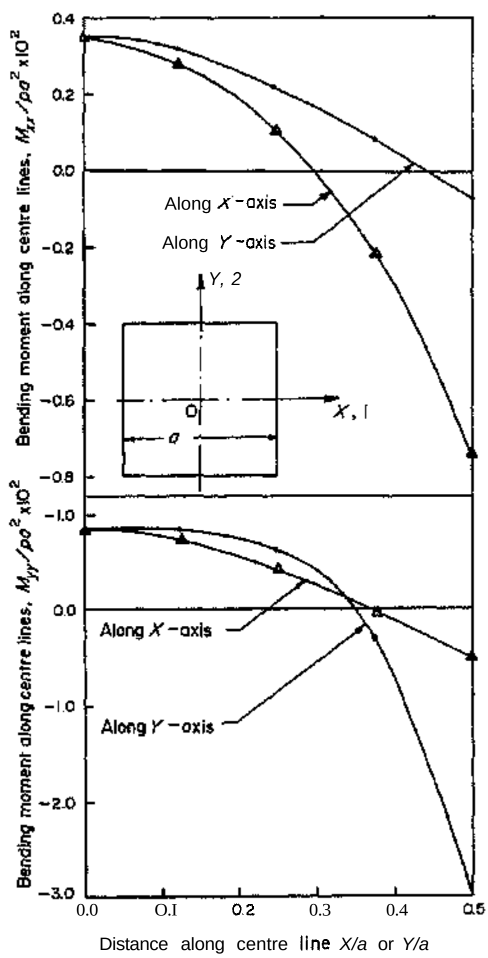


Fig. 8. Moment distributions along centre lines of clamped glass-epoxy composite square plate.

and deflections is not possible due to inherent approximations in the available solution in Ref. [13]. (The soluion in [13] when specialized to isotropic case shows an error of about 6.3% for maximum deflection and unacceptable accuracy for maximum moments.)

An orthotropic cylindrical shell with a circular hole subjected to axial tension. The problem illustrated in Fig. 9 is analysed to evaluate the coupled membrane-bending behaviour of the cylindrical shell form of the element. Taking advantage of symmetry, only one quarter of the shell needs to be analysed. Choosing the 5 x 5 finite element grid illustrated in Fig. 3, 51 elements with 37 nodes (367 degrees of freedom) are used to model the quarter of the shell. The adequacy of the chosen grid to solve the present problem was checked by solving an otherwise identical isotropic shell problem. Typical results are given in Table 5 along with continuum solutions [15] for comparison to demonstrate the accuracy achieved with the rather coarse mesh used.

Predicted stress distributions for both isotropic and orthotropic shells are shown in Figs. 10 and 11 along With the results of anisotropic elasticity solution to the problem of an infinite plate with a circular hole [11]. The predicted stresss distributions are fairly smooth and appear to have converged as well. Near the hole the coupled membrane-bending behaviour appears to be

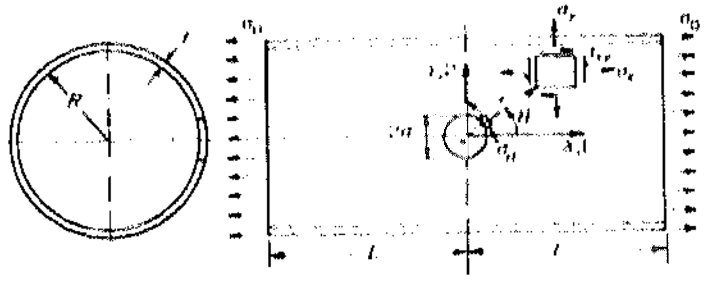
Table 4. Clamped glass-epoxy composite square plate subjected to uniform pressure load-comparison of results

	Maximum deflection WD_{11}/pa^4	Bending moments at $X = a/2$, $Y = 0$ M_{xx}/pa^2	M_{yy}/pa^2	Bending moments at $X = 0$, $Y = a/2$ M_{xx}/pa^2	M_{yy}/pa^2
Present results	0.00216	−0.0744	0.00495	0.00732	0.02938
Approximate analytical solution Ref. [10]	0.002313	0.06517	0.004333	0.00433	−0.01732

Table 5 Cylindrical shell with a circular hole under axial tension Comparison of results

	Membrane stress ratio $(\sigma_\theta^m/\sigma_0)$ at $\theta = 10°$	at $\theta = 90°$	Bending stress† ratio $(\sigma_\theta^b/\sigma_0)$ at $\theta = 0$	at $\theta = 90°$
Orthotropic, shell Present results	0.503648	4.9389	0.301821	0.4391
Orthotropic plate Solution Ref. [12]	0.4813	4.8963	0.0	0.0
Isotropic shell Present results	−1.26018	3.70422	0.61575	0.5203
Isotropic shell results from Ref. [11]	1.25	3.6	0.80	0.45

†At inner surface of the shell



GEOMETRIC PARAMETERS		MATERIAL CONSTANTS ISOTROPIC (Aluminium)	ORTHOTROPIC Glass Epoxy composite
$R/t = 100$	E_{xx}	[illegible] psi	[illegible] psi
$a/t = 10$	$E_{\theta\theta}$	[illegible] psi	[illegible] psi
$L/R = 5$	$G_{x\theta}$	[illegible] psi	[illegible] psi
$\theta = \pm 1.0$ [illegible]	$\nu_{x\theta}$	[illegible]	[illegible]
	$\nu_{\theta x}$	[illegible]	[illegible]

Fig. 9. Cylindrical shell with a circular hole under axial tension.

characterized adequately by the small number of elements used. Predicted membrane and bending stress concentration factors are presented in Table 1 The main difficulty in assessing accuracy of these results arises from the fact that authors coud not find any other result for the orthotropic shell for comparison. Since the isotropic shell is a particular case of the orthotropic shell treated here it might be expected that the same order of accuracy holds good for both.

CONCLUSIONS

The stiffness matrix for a high precision triangular [illegible] been formulated, and coded into a a posite structural analysis program The versatility of e element's formulation enables its use in the analy ; of multilayered composite plate and cylindrical shel structures taking into account actual lamination paran ters. The results obtained with the use of the sub] :t finite element demonstrated good engineering accu .cy for stress as well as displacements in the example plications with a modest number of elements.

The ultimate objective of any ana sis is as a design toot This work implies the iterative ie of the analysis procedures in a design exercise. The present finite element formulation is ideally suited 1 · computer aided design of composite structures.

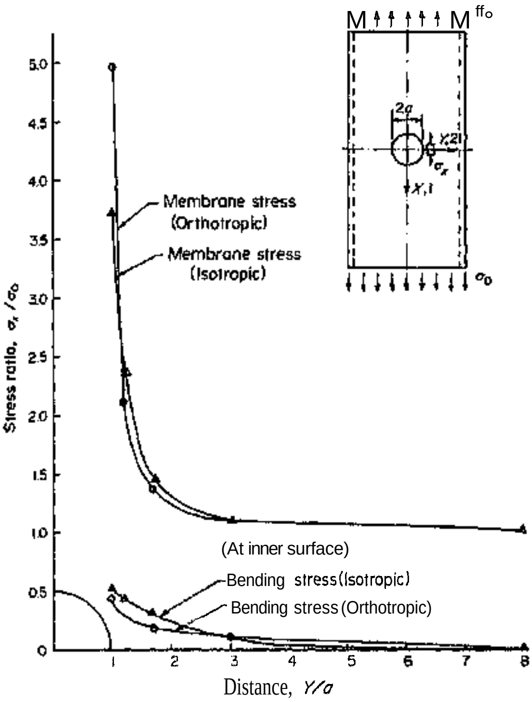
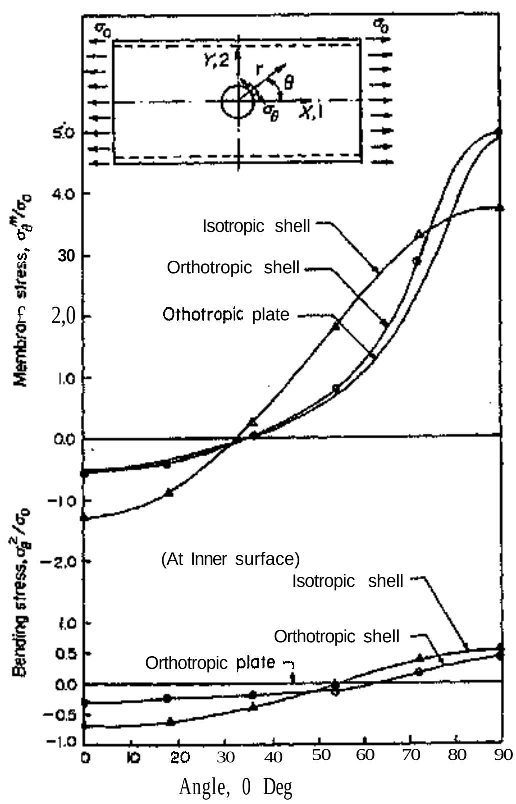


Fig. 10. Axial stress distribution along Y-axis.



Pig. 11. Tangential stress distribution along the hole boundary.

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APPENDIX

Stiffness Matrix *[k]*

$$
\begin{aligned}
k_{ij} = {} & \bar{L}_{11} m_i m_j F(m_i + m_j - 2, n_i + n_j) \\
j \geq i & + \bar{L}_{12}(m_i n_j + m_j n_i) F(m_i + m_j - 1, n_i + n_j - 1) \\
& + \bar{L}_{13} m_i p_j F(m_i + p_j - 2, n_i + q_j) \\
& + \bar{L}_{14} m_i q_j F(m_i + p_j - 1, n_i + q_j - 1) \\
& + \bar{L}_{15} m_i F(m_i + r_j - 1, n_i + s_j) G_j \\
& + \bar{L}_{16} m_i r_j (r_j - 1) F(m_i + r_j - 3, n_i + s_j) \\
& + \bar{L}_{17} m_i r_j s_j F(m_i + r_j - 2, n_i + s_j - 1) \\
& + \bar{L}_{18} m_i s_j (s_j - 1) F(m_i + r_j - 1, n_i + s_j - 2) \\
& + \bar{L}_{22} n_i n_j F(m_i + m_j, n_i + n_j - 2) \\
& + \bar{L}_{23} n_i p_j F(m_i + p_j - 1, n_i + q_j - 1) \\
& + \bar{L}_{24} n_i q_j F(m_i + p_j, n_i + q_j - 2) \\
& + \bar{L}_{25} n_i F(m_i + r_j, n_i + s_j - 1) G_j \\
& + \bar{L}_{26} n_i r_j (r_j - 1) F(m_i + r_j - 2, n_i + s_j - 1) \\
& + \bar{L}_{27} n_i r_j s_j F(m_i + r_j - 1, n_i + s_j - 2) \\
& + \bar{L}_{28} n_i s_j (s_j - 1) F(m_i + \mathrm{fy}\ n_i + s_j - 3) \\
& + \bar{L}_{33} p_i p_j F(p_i + p_j - 2, q_i + q_j) \\
& + \bar{L}_{34}(p_i q_j + p_j q_i) F(p_i + p_j - 1, q, + q_j - 1) \\
& + \bar{L}_{35} p_i F(p_i + r_j - 1, q_i + s_j) G_j \\
& + \bar{L}_{36} p_i r_j (r_j - 1) F(p_i + r_j - 3, q_i + s_j) \\
& + \bar{L}_{37} p_i r_j s_j F(p_i + r_j - 2, q_i + s_j - 1) \\
& + \bar{L}_{38} p_i s_j s_{j-1} F(p_i + r_j - 1, q, + s_j - 2) \\
& + \bar{L}_{44} q_i q_j F(p_i + p_j, q_i + q_j - 2) \\
& + \bar{L}_{45} q_i F(p_i + r_j, q_i + s_{j-1}) G_j \\
& + \bar{L}_{46} q_i r_j (r_{j-1}) F(p_i + r_j - 2, q_i + s_j - 1) \\
& + \bar{L}_{47} q_i r_j s_j F(p_i + r_j - 1, q_i + s_j - 2)
\end{aligned}
$$

$$
\begin{aligned}
&+\bar{L}_{48}q_is_js_{j-1}F(p_i+r_j,q_i+s_j-3)\\
&+\bar{L}_{55}F(r_i+r_j,S_i+s_j)G_iG_j\\
&+\bar{L}_{56}r_j(r_j-1)F(r_i+r_j-2,s_i+s_j)G_i\\
&+\bar{L}_{57}r_js_jF(r_i+r_j-1,s_i+s_j-1)G_i\\
&+\bar{L}_{58}s_j(s_j-1)F(r_i+r_j,s_i+s_j-2)G_i\\
&+\bar{L}_{66}r_ir_j(r_i-1)(r_j-1)F(r_i+r_j-4,s_i+s_j)\\
&+\bar{L}_{67}r_ir_j(r_i-1)s_jF(r_i+r_j-3,s_i+s_j-1)\\
&+\bar{L}_{68}[r_is_j(r_i-1)(s_j-1)+r_js_i(r_j-1)(s_i-1)]\\
&\times F(r_i+r_j-2,s_i+s_j-2)\\
&+\bar{L}_{77}r_ir_js_is_jF(r_i+r_j-2,s_i+s_j-2)\\
&+\bar{L}_{78}r_is_is_j(s_j-1)F(r_i+r_j-1,s_i+s_j-3)\\
&+\bar{L}_{88}s_is_j(s_i-1)(s_j-1)F(r_i+r_j,s_i+s_j-4)\\
&k_{ij}
\end{aligned}
$$

where

$$F(m,n)=(-1)^{n+1}[a^{m+1}-(-b)^{m+1}]c^{n+1}\frac{m!n!}{(m+n+2)!}$$

$$m_i,n_i=0 \quad \text{for} \quad i\le 10$$

$$p_i,q_i=0 \quad \text{for} \quad 11\le i\le 20$$

$$r_i,s_i=0 \quad \text{for} \quad i\le 20$$

$$G_i=0 \quad \text{for} \quad i\le 20,$$

$$G_i=1 \quad \text{for} \quad i>20.$$